

# Kontsevich formality and PBW algebras

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*Ты пой, моя радость, ты пой, моё чудо, гитара,  
Прозрачные струны случайно счастливого мира...*

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## Abstract

Let  $\alpha$  be a polynomial Poisson bivector on a finite-dimensional vector space  $V$  over  $\mathbb{C}$ . Then Kontsevich [K97] gives a formula for a quantization  $f \star g$  of the algebra  $S(V)^*$ . We give a construction of an algebra with the PBW property defined from  $\alpha$  by generators and relations. Namely, we define an algebra as the quotient of the free tensor algebra  $T(V^*)$  by relations  $x_i \otimes x_j - x_j \otimes x_i = R_{ij}(\hbar)$  where  $R_{ij}(\hbar) \in T(V^*) \otimes \hbar\mathbb{C}[[\hbar]]$ ,  $R_{ij} = \hbar \text{Sym}(\alpha_{ij}) + \mathcal{O}(\hbar^2)$ , with one relation for each pair of  $i, j = 1 \dots \dim V$ . We prove that the constructed algebra obeys the PBW property, and this is a generalization of the Poincaré-Birkhoff-Witt theorem. In the case of a linear Poisson structure we get a new very conceptual proof of the PBW theorem itself, and for a quadratic Poisson structure we get an object closely related to a quantum  $R$ -matrix on  $V$ .

The construction uses the Kontsevich formality. Namely, our quantities  $R_{ij}(\hbar) \in T(V^*) \otimes \mathbb{C}[[\hbar]]$  are written directly in Kontsevich integrals from [K97], but in a sense of dual graphs than the graphs used in the deformation quantization. We conjecture that the relation  $x_i \otimes x_j - x_j \otimes x_i = R_{ij}(\hbar)$  holds in the Kontsevich star-algebra, when we replace  $\otimes$  by  $\star$ . This conjecture implies in particular that our algebra is isomorphic to the Kontsevich star-algebra with the same  $\alpha$ , but also gives a highly-nontrivial identity on Kontsevich integrals. Probably, it is a particular case of a more general duality acting on the AKSZ model on open disc, used by Kontsevich in his proof of the formality conjecture.

## Introduction

### 0.1

Let  $\alpha$  be a polynomial Poisson bivector on a vector space  $V$ . There are two ways how one can think about what is to quantize  $\alpha$ .

The first way, called deformation quantization, is well-known. One looks for an associative product on the space  $C^\infty(V)$  of smooth functions on  $V$  having a form

$$f \star g = f \cdot g + \hbar \frac{1}{2} \{f, g\}_\alpha + \hbar^2 B_2(f, g) + \hbar^3 B_3(f, g) + \dots \quad (1)$$

where  $\{f, g\}_\alpha = \alpha(df \wedge dg)$  is the corresponding Poisson bracket, and  $B_i: C^\infty(V)^{\otimes 2} \rightarrow C^\infty(V)$  are some local maps, which in practice are bi-differential operators. The associativity condition

$$(f \star g) \star h = f \star (g \star h) \quad (2)$$

for any  $f, g, h \in C^\infty(V)$  is then an infinite sequence of quadratic identities on  $B_i$ 's. This problem was solved by M.Kontsevich in 1997, and the solution uses a two-dimensional topological quantum field theory on open disc.

There is another way how to think about the "quantization". Namely, we are looking for an algebra which is a quotient of the tensor algebra  $T(V^*)$  by relations of the form

$$x_i \otimes x_j - x_j \otimes x_i = \hbar \text{Sym}(\alpha_{ij}) + \hbar^2 \omega_2 + \hbar^3 \omega_3 + \dots \quad (3)$$

Here  $\{x_i\}$  is a basis in  $V^*$ , and  $\alpha = \sum_{i,j} \alpha_{ij} \partial_i \wedge \partial_j$ , where  $\alpha_{ij}$  are polynomials. The symmetrization  $\text{Sym}(\alpha_{ij})$  is the element of  $T(x_1, \dots, x_n)$  which is given by the full symmetrization,  $\text{Sym}: \mathbb{C}[x_1, \dots, x_n] \rightarrow T(x_1, \dots, x_n)$ . Here  $\omega_2, \omega_3, \dots$  are some elements of  $T(x_1, \dots, x_n)$ , they depend on the pair  $(i, j)$  of indices. So, the problem is to find such  $\omega_k(i, j)$ 's such that the obtained algebra obeys the Poincaré-Birkhoff-Witt (PBW) property. Let us formulate this property.

Denote by  $A_\hbar$  the algebra over  $\mathbb{C}[[\hbar]]$  given by (3). We have the following decreasing filtration of  $A_\hbar$ :

$$A_\hbar \supset \hbar A_\hbar \supset \hbar^2 A_\hbar \supset \dots \quad (4)$$

It is an algebra filtration, that is,  $(\hbar^i A_\hbar) \cdot (\hbar^j A_\hbar) \subset \hbar^{i+j} A_\hbar$ . Consider the associated graded algebra  $\text{gr} A_\hbar = \bigoplus_{k \geq 0} \hbar^k A_\hbar / \hbar^{k+1} A_\hbar$ . The PBW condition then is that there is an  $\hbar$ -linear isomorphism  $\text{gr} A_\hbar \simeq S(V^*) \otimes \mathbb{C}[[\hbar]]$  with which  $\hbar^k A_\hbar / \hbar^{k+1} A_\hbar \simeq \hbar^k S(V^*)$ .

If the algebra  $A_\hbar$  is given by (3), the condition above always holds for  $k = 0$ , and the only condition which is necessary for  $k = 1$  is  $\alpha_{ij} = -\alpha_{ji}$ . For  $k \geq 2$  in general the component  $\hbar^k A_\hbar / \hbar^{k+1} A_\hbar$  is *less* than  $\hbar^k S(V^*)$ . We can go one step further and find the necessary condition for  $k = 2$ .

Consider  $\text{Alt}_{i,j,k}[x_i, [x_j, x_k]]$  where  $[a, b] = a \star b - b \star a$ . This expression is 0 for any associative algebra. Actually this condition is an infinite sequence of conditions, but its image in  $\hbar^2 A_\hbar / \hbar^3 A_\hbar$  depends only on  $\{\alpha_{ij}\}$ . It is exactly the condition  $\{\alpha, \alpha\} = 0$  where the bracket is the Schouten-Nijenhuis bracket. If this condition is not satisfied, we get a nonzero element in  $\hbar^2 S(V^*)$  which is zero in  $\hbar^2 A_\hbar / \hbar^3 A_\hbar$ . One can check that if the Schouten-Nijenhuis bracket  $\{\alpha, \alpha\}$  is zero, then  $\hbar^2 A_\hbar / \hbar^3 A_\hbar \simeq \hbar^2 S(V^*)$ .

For higher orders in  $\hbar$  we have more complicated conditions. The claim is that if  $\alpha$  is a Poisson bivector, we can find all  $\omega_k(i, j)$  in (3) such that the algebra defined by (3) is a PBW algebra. The reader can see from this discussion that this question is very close to the classical question of deformation quantization.

## 0.2

The Kontsevich deformation quantization formula gives us a PBW algebra associated with a Poisson bivector  $\alpha$ . Indeed, we have some formula like  $x_i \star x_j = x_i \cdot x_j + \frac{1}{2}\hbar\alpha_{ij}(x_1, \dots, x_n) + \hbar^2(\dots) + \dots$ . Here  $\star$  is the Kontsevich star product. We see that  $x_i \star x_j - x_j \star x_i$  starts with the first order in  $\hbar$ . The right-hand side is an element in  $S(V^*) \otimes \hbar\mathbb{C}[[\hbar]]$ . If we express iteratively the right-hand side as sum of monomials of the form  $x_{i_1} \star x_{i_2} \star \dots \star x_{i_k}$ , we can then replace  $\star$  by  $\otimes$  and get a PBW algebra. It will be indeed a PBW algebra, because the associated graded algebra has at most the size as  $S(V^*) \otimes \mathbb{C}[[\hbar]]$ , and it can not have a less size because this relation holds in the Kontsevich algebra. This proves in particular that one doesn't need any other relation to define the Kontsevich star-algebra.

A lack of this construction is that we apply Kontsevich formula, or rather something like a reverse to it, infinitely many times. The coefficients will depend on the Kontsevich integrals in deformation quantization, but actually will be much more complicated.

Our solution uses the Kontsevich integrals of, in a sense, dual admissible graphs. In particular, they are given directly in Kontsevich integrals, without any iterative process, but of dual graphs. We conjecture that our relation holds exactly in the Kontsevich star-algebra. If this conjecture is true, we get a very complicated relation between Kontsevich integrals.

## 0.3

It is clear that if the conjecture described in the previous Subsection is true, it should have an analogue for all integrals, not only for particular graphs involving in the deformation quantization formula. We say that we want to lift this conjecture "on the level of complexes". Moreover, we believe that some "Koszul duality" acts on the entire AKSZ model on open disc, and we want to express this duality mathematically. We are going to consider this question in the sequel.

## 0.4

All quadratic PBW algebras are Koszul. One easily sees that if one starts with a quadratic Poisson bivector  $\alpha$ , the relations we get are quadratic, namely, all  $\omega_k(i, j)$  are elements in  $V^* \otimes V^* \subset T(V^*)$ . The dg Lie algebra of polyvector fields on  $V$  is isomorphic to the dg Lie algebra of polyvector fields on  $V^*[1]$ , and this isomorphism preserves quadratic bivector fields. Then, our constructions give two PBW algebras from a quadratic Poisson

bivector on  $V$ : one is the quotient of  $T(V^*)$  by some quadratic relations, and another is a quotient of  $T(V[-1])$  by some other quadratic relations. It is natural to conjecture that the two algebras are Koszul dual. This explains our notation "Koszul duality in deformation quantization". We are going to consider these questions in a sequel paper.

## 0.5

We would like especially to note, that the  $L_\infty$  map

$$\Theta: T_{poly}(V) \rightarrow \text{Der}(CoBar^\bullet(\Lambda^-(V^*))) / \text{Inn}(CoBar^\bullet(\Lambda^-(V^*))) \quad (5)$$

which is the composition of the map  $\Phi$  from Section 1.4 with the Kontsevich formality  $L_\infty$  map, can be considered as a "non-commutative analogue" of the usual map (actually an isomorphism)  $W_n \rightarrow \text{Der}(S(V^*))$  where  $W_n$  is the Lie algebra of polynomial vector fields on  $V$ . Probably it is possible to define some "quasi-manifolds" by replacing the local coordinate ring  $S(V^*)$  by its free resolution  $CoBar^\bullet(\Lambda^-(V^*))$ , and by replacing the usual transition functions by  $A_\infty$  quasi-isomorphisms with the natural compatibility property. We need to invert some of these  $A_\infty$  quasi-isomorphisms in the compatibility equation on the "triple intersection"  $U_i \cap U_j \cap U_k$ , therefore, we should work in the Quillen homotopical category. Any usual manifold gives us a trivial example of a quasi-manifold by means of the canonical projection of the resolution to the algebra. The corresponding Lie algebra of infinitesimal symmetries in the sense of the formal geometry will be then  $T_{poly}(V)$ , by means of the quasi-isomorphism (5).

A first difficulty in the realization of this program is that our map (5), and even the resolution, are defined for polynomial algebras, not algebras of smooth functions. Nevertheless, the author is sure that an algebraic analog of this construction, leading to a rigorous definition of a "quasi-manifold", should exist. We hope to clarify it in the sequel.

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## 1 The main construction

### 1.1

First of all, recall here the Stasheff's definition of the Hochschild cohomological complex of an associative algebra  $A$ .

Consider the shifted vector space  $W = A[1]$ , and the cofree coassociative coalgebra  $C(W)$  (co)generated by  $W$ . As a graded vector space,  $C(W) = T(A[1])$ , the free tensor space. The coproduct is:

$$\Delta(a_1 \otimes a_2 \otimes \cdots \otimes a_k) = \sum_{i=1}^{k-1} (a_1 \otimes \cdots \otimes a_i) \bigotimes (a_{i+1} \otimes \cdots \otimes a_k) \quad (6)$$

Consider the Lie algebra  $CoDer(C(A[1]))$  of all coderivations of this coalgebra. As the coalgebra is free, any coderivation  $D$  (if it is graded) is uniquely defined by a map  $\Psi_D: A^{\otimes k} \rightarrow A$ , and the degree of this coderivation is  $k - 1$  (in conditions that  $A$  is not graded). The bracket  $[\Psi_{D_1}, \Psi_{D_2}]$  is again a coderivation. Define the Hochschild Lie algebra as  $Hoch^\bullet(A) = CoDer^\bullet(C(A[1]))$ . To define the complex structure on it, consider the particular coderivation  $D_m$  of degree  $+1$  from the product  $m: A^{\otimes 2} \rightarrow A$ , which is the product in the associative algebra  $A$ . The condition  $[D_m, D_m] = 0$  is equivalent to the associativity of  $m$ . Define the differential on  $CoDer^\bullet(C(A[1]))$  as  $d(\Psi) = [D_m, \Psi]$ . In this way we get a dg Lie algebra. The differential is called the Hochschild differential, and the bracket is called the Gerstenhaber bracket. The definition of these structures given here is due to J.Stasheff.

## 1.2 The explicit definition

Here we relate the Stasheff's definition of the Hochschild cohomological complex with the usual one.

The concept of a coderivation of a (co)free coalgebra is dual to the concept of a derivation of a free algebra. Let  $L$  be a vector space, and let  $T(L)$  be the free tensor algebra generated by the vector space  $L$ . Let  $D: T(L) \rightarrow T(L)$  be a derivation, then it is uniquely defined by its value  $D_L: L \rightarrow T(L)$  on the generators, and any  $D_L$  defines a derivation  $D$  of the free algebra  $T(L)$ . If we would like to consider only graded derivations, we restrict ourselves by the maps  $D_L: L \rightarrow L^{\otimes k}$  for  $k \geq 0$ .

Dually, a coderivation  $D$  of the cofree coalgebra  $C(P)$  cogenerated by a vector space  $P$  is uniquely defined by the restriction to cogenerators, that is, by a map  $D_P: C(P) \rightarrow P$ , or, if we consider the graded coderivations, the map  $D_P$  is a map  $D_P: P^{\otimes k} \rightarrow P$  for  $k \geq 0$ .

In our case of the definition of the cohomological Hochschild complex of an associative algebra  $A$ , we have  $P = A[1]$ . Then the coderivations of the grading  $k$  form the vector space  $Hoch^k(A) = Hom(A^{\otimes(k+1)}, A)$ ,  $k \geq -1$ . Now we can deduce the differential and the Gerstenhaber bracket from the Stasheff's construction. The answer is the following:

For  $\Psi \in Hom(A^{\otimes k}, A)$  the cochain  $d\Psi \in Hom(A^{\otimes(k+1)}, A)$  is given by the formula:

$$\begin{aligned}
d\Psi(a_0 \otimes \cdots \otimes a_k) &= a_0 \Psi(a_1 \otimes \cdots \otimes a_k) + \\
&+ \sum_{i=0}^{k-1} (-1)^{i+1} \Psi(a_0 \otimes \cdots \otimes a_{i-1} \otimes (a_i a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_k) + \\
&+ (-1)^{k+1} \Psi(a_0 \otimes \cdots \otimes a_{k-1}) a_k
\end{aligned} \tag{7}$$

For  $\Psi_1 \in \text{Hom}(A^{\otimes(k+1)}, A)$  and  $\Psi_2 \in \text{Hom}(A^{\otimes(l+1)}, A)$  the bracket  $[\Psi_1, \Psi_2] = \Psi_1 \circ \Psi_2 - (-1)^{kl} \Psi_2 \circ \Psi_1$  where

$$\begin{aligned}
(\Psi_1 \circ \Psi_2)(a_0 \otimes \cdots \otimes a_{k+l}) &= \\
\sum_{i=0}^k (-1)^{il} \Psi_1(a_0 \otimes \cdots \otimes a_{i-1} \otimes \Psi_2(a_i \otimes \cdots \otimes a_{i+l}) \otimes a_{i+l+1} \otimes \cdots \otimes a_{k+l})
\end{aligned} \tag{8}$$

### 1.3 The (co)bar-complex

Here we recall the definition of the (co)bar-complex of an associative (co)algebra. When the (co)algebra contains (co)unit, the (co)bar-complex is acyclic, and when the (co)algebra is the kernel of the augmentation of a quadratic Koszul algebra, this concept is closely related to the Koszul duality.

Let  $A$  be an associative algebra. Then its bar-complex is

$$\cdots \rightarrow A^{\otimes 3} \rightarrow A^{\otimes 2} \rightarrow A \rightarrow 0$$

where  $\deg A^{\otimes k} = -k + 1$ , and the differential  $d: A^{\otimes k} \rightarrow A^{\otimes(k-1)}$  is given as follows:

$$d(a_1 \otimes \cdots \otimes a_k) = (a_1 a_2) \otimes a_3 \otimes \cdots \otimes a_k - a_1 \otimes (a_2 a_3) \otimes \cdots \otimes a_k + \cdots + (-1)^k a_1 \otimes \cdots \otimes a_{k-2} \otimes (a_{k-1} a_k) \tag{9}$$

If the algebra  $A$  has unit, the bar-complex of  $A$  is acyclic in all degrees. Indeed, the map

$$a_1 \otimes \cdots \otimes a_k \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_k$$

is a contracting homotopy.

Suppose now that the algebra  $A$  does not contain unit, and  $A = B^+$  is the kernel of an augmentation map  $\varepsilon: B \rightarrow \mathbb{C}$ . (The map  $\varepsilon$  is a surjective map of algebras, in particular, it maps 1 to 1). Then the cohomology of the bar-complex of  $A$  is equal to the dual space  $\text{Ext}_{B\text{-Mod}}^*(\mathbb{C}, \mathbb{C})$ .

Indeed, for any  $B$ -module  $M$ , we have the following free resolution of  $M$ :

$$\cdots B \otimes \overline{B} \otimes \overline{B} \otimes M \rightarrow B \otimes \overline{B} \otimes M \rightarrow B \otimes M \rightarrow M \rightarrow 0 \tag{10}$$

with the differential analogous to the bar-differential.

Consider the case  $M = \mathbb{C}$ . We can compute  $\text{Ext}_{B\text{-}Mod}^\bullet(\mathbb{C}, \mathbb{C})$  using this resolution. In the answer we get the cohomology of the complex dual to the bar-complex of  $A = B^+$ .

The complex dual to the bar-complex of  $A$  is the *cobar-complex* for the coalgebra  $A^*$ . This cobar-complex is an associative dg algebra, and it is a free algebra, which by previous is a free resolution of the algebra  $\text{Ext}_{B\text{-}Mod}(\mathbb{C}, \mathbb{C})$ . For the sequel we write down explicitly the cobar-complex of a coassociative coalgebra  $Q$ :

$$0 \rightarrow Q \rightarrow Q \otimes Q \rightarrow Q \otimes Q \otimes Q \rightarrow \dots \quad (11)$$

and the differential  $\delta Q^{\otimes k} \rightarrow Q^{\otimes(k+1)}$  is

$$\delta(q_1 \otimes \dots \otimes q_k) = (\Delta q_1) \otimes q_2 \otimes \dots \otimes q_k - q_1 \otimes (\Delta q_2) \otimes \dots \otimes q_k + \dots + (-1)^{k-1} q_1 \otimes \dots \otimes q_{k-1} \otimes (\Delta q_k) \quad (12)$$

where  $\Delta: Q \rightarrow Q^{\otimes 2}$  is the coproduct.

In the case when  $B = S(V)$  is the symmetric algebra,  $\text{Ext}_{B\text{-}Mod}(\mathbb{C}, \mathbb{C})$  is the exterior algebra  $\Lambda(V^*) = S(V^*[-1])$ , and vice versa. In this way, we get a free resolution of the symmetric (exterior) algebra.

*Example.* Here we construct explicitly the free cobar-resolution  $\mathcal{R}^\bullet$  of the algebra  $\mathbb{C}[x_1, x_2]$  of polynomials on two variables. As a graded algebra,  $\mathcal{R}^\bullet$  is the free algebra  $\mathcal{R}^\bullet = \text{Free}(x_1, x_2, \xi_{12})$  where  $\deg x_1 = \deg x_2 = 0$ ,  $\deg \xi_{12} = -1$ . The differential is 0 on  $x_1, x_2$ ,  $d(\xi_{12}) = x_1 \otimes x_2 - x_2 \otimes x_1$ , and satisfies the graded Leibniz rule. In degree 0 we have the tensor algebra  $T(x_1, x_2)$ , differential is 0 on degree 0 (there are no elements in degree 1). In degree -1, a general element is a non-commutative word in  $x_1, x_2, \xi_{12}$  in which  $\xi_{12}$  occurs exactly one time. For example, it could be a word  $x_2 \otimes x_1 \otimes x_2 \otimes \xi_{12} \otimes x_1 \otimes x_2$ . The image of the differential is then exactly the two-sided ideal in the tensor algebra  $T(x_1, x_2)$  generated by  $x_1 \otimes x_2 - x_2 \otimes x_1$ . Then, the 0-th cohomology is  $\mathbb{C}[x_1, x_2]$ . It follows from the discussion above that all higher cohomology is 0.

*Example.* Consider the case of the algebra  $\mathbb{C}[x_1, x_2, x_3]$ . Then  $\mathcal{R}^\bullet = \text{Free}(x_1, x_2, x_3, \xi_{12}, \xi_{23}, \xi_{13}, \xi_{123})$  with  $\deg x_i = 0$ ,  $\deg \xi_{ij} = -1$ ,  $\deg \xi_{123} = -2$ . The differential is 0 on  $x_1, x_2, x_3$ ,  $d(\xi_{ij}) = x_i \otimes x_j - x_j \otimes x_i$ , and  $d(\xi_{123}) = (x_1 \otimes \xi_{23} + x_2 \otimes \xi_{31} + x_3 \otimes \xi_{12}) + (\xi_{23} \otimes x_1 + \xi_{31} \otimes x_2 + \xi_{12} \otimes x_3)$ . Here we set  $\xi_{ij} = -\xi_{ji}$ . Then cohomology in degree 0 is equal to the quotient of the free algebra  $T(x_1, x_2, x_3)$  by the two-sided ideal generated by  $x_i \otimes x_j - x_j \otimes x_i$ , that is, the algebra  $\mathbb{C}[x_1, x_2, x_3]$ . It follows from our general discussion in this Subsection that the higher cohomology vanishes.

## 1.4 The main construction

Here we construct a quasi-isomorphic map of dg Lie algebras  $\Phi: \text{Hoch}^\bullet(S(V))/\mathbb{C} \rightarrow \text{Der}(\text{CoBar}^\bullet(S(V^*)^+)/\text{Inn}(\text{CoBar}^\bullet(S(V^*)^+)))$ .

Let  $\Psi \in \text{Hom}((S(V))^{\otimes k}, S(V))$  be a  $k$ -cochain. Denote  $V^* = W$ , then we can consider  $\Psi$  the corresponding cochain in  $\text{Hom}(S(W), (S(W))^{\otimes k})$ . Here we consider  $S(W)$  as *coalgebra*. Then this cochain may be considered as a derivation in  $\text{Der}(\text{CoBar}^\bullet(S(W)))$ . We

would like to attach to it a derivation in  $\text{Der}(\text{CoBar}^\bullet(S(W)^+))$ , maybe modulo an inner derivation. So, we would like to show that there exist a map  $\Phi: \text{Der}(\text{CoBar}^\bullet(S(W))) \rightarrow \text{Der}(\text{CoBar}^\bullet(S(W)^+))$  such that the diagram

$$\begin{array}{ccc} \text{Der}(\text{CoBar}^\bullet(S(W))) & \xrightarrow{\delta} & \text{Der}(\text{CoBar}^\bullet(S(W))) \\ \downarrow \Phi & & \downarrow \Phi \\ \text{Der}(\text{CoBar}^\bullet(S(W)^+)) & \xrightarrow{\delta} & \text{Der}(\text{CoBar}^\bullet(S(W)^+)) \end{array} \quad (13)$$

is commutative *modulo inner derivations* (here  $\delta$  is the cobar differential).

In the coalgebra  $S(W)$  the coproduct is given by the formula

$$\Delta(x_1 \dots x_k) = 1 \otimes (x_1 \dots x_k) + \sum_{\{i_1 \dots i_a\} \sqcup \{j_1 \dots j_b\} = \{1 \dots k\}} (x_{i_1} \dots x_{i_a}) \otimes (x_{j_1} \dots x_{j_b}) + (x_1 \dots x_k) \otimes 1 \quad (14)$$

and in the coalgebra  $S(W)^+$  the coproduct is given by the same formula without the first and the last summands, which contain 1's.

Therefore the projection  $p: S(W) \rightarrow S(W)^+$  is a map of coalgebras (dual to the imbedding of algebras), and the imbedding  $i: S(W)^+ \rightarrow S(W)$  is *not*.

If  $\Psi: S(W) \rightarrow S(W)^{\otimes k}$  is as above, we define  $(\Phi(\Psi))(\sigma) = p^{\otimes k}(\Psi(i(\sigma))) \in \text{Hom}(S(W)^+, (S(W)^+)^{\otimes k})$ .

Now we check the commutativity of the diagram (13) modulo inner derivations. It is clear that

$$(\Phi \circ \delta)(\sigma) - (\delta \circ \Phi)(\sigma) = p^{\otimes k}(\Psi(1)) \otimes \sigma \pm \sigma \otimes p^{\otimes k}(\Psi(1)) \quad (15)$$

which is an inner derivation  $ad(p^{\otimes k}(\Psi(1)))$ .

We have defined a map  $\Phi_1: \text{Der}(\text{CoBar}^\bullet(S(W))) \rightarrow \text{Der}(\text{CoBar}^\bullet(S(W)^+))/\text{Inn}(\text{CoBar}^\bullet(S(W)^+))$ . The first dg Lie algebra is clearly isomorphic to the Hochschild cohomological complex of the algebra  $S(V)$  modulo constants, and we can consider the map  $\Phi_1$  as a map

$$\Phi_1: \text{Hoch}^\bullet(S(V))/\mathbb{C} \rightarrow \text{Der}(\text{CoBar}^\bullet(S(V^*)^+))/\text{Inn}(\text{CoBar}^\bullet(S(V^*)^+))$$

Let us note that the cobar complex  $\text{CoBar}^\bullet(S(V^*)^+)$  is a free resolution of the Koszul dual algebra  $\Lambda(V^*)$ .

Now we have the following result:

**Proposition.** *The map  $\Phi_1$  is a quasi-isomorphism of dg Lie algebras.*

It is clear that  $\Phi_1$  is a map of dg Lie algebras, one only needs to proof that it is a quasi-isomorphism of complexes. We will not use this result, the proof will appear somewhere. Let us, however, outline some naive ideas behind the proof in the next Subsection.



## 1.5

We prove firstly that the cohomology of  $\text{Der}(CoBar^\bullet(S(V^*)^+))/\text{Inn}(CoBar^\bullet(S(V^*)^+))$  is isomorphic to the Hochschild cohomology  $\text{Hoch}^\bullet(CoBar^\bullet(S(V^*)^+))$  of the dg algebra  $CoBar^\bullet(S(V^*)^+)$ . Let us note that it is not true that the Hochschild cohomology  $\text{Hoch}^\bullet(CoBar^\bullet(B))/\mathbb{C}$  is isomorphic to the cohomology of  $\text{Der}(CoBar^\bullet(B))/\text{Inn}(CoBar^\bullet(B))$  for any coassociative coalgebra  $B$ . To have this property,  $B$  should be *cocomplete*, that is

$$B = \bigcup_{n \geq 1} \text{Ker}(\Delta^n: B \rightarrow B^{\otimes n+1}) \quad (16)$$

This fact is proven in [Lef]. The coalgebras  $S^+(V)$ ,  $\Lambda^-(V)$  are clearly cocomplete for any vector space  $V$ . The coalgebra  $S(V)$  is not cocomplete, and the property fails for it. Indeed, let us suppose that  $\text{Der}(CoBar^\bullet(S(V)))/\text{Inn}(CoBar^\bullet(S(V)))$  has the same cohomology that  $\text{Hoch}^\bullet(CoBar^\bullet(S(V)))$ . The dg algebra  $CoBar^\bullet(S(V))$  is acyclic, and the Hochschild cohomology of quasi-isomorphic algebras are the same; we conclude, that  $HH^\bullet(CoBar^\bullet(S(V))) = 0$ . But the cohomology of  $\text{Der}^\bullet(CoBar^\bullet(S(V)))/\text{Inn}(CoBar^\bullet(S(V)))$  is equal to polyvector fields  $T_{poly}(V)/\mathbb{C}$ . Indeed, consider the short exact sequence of complexes

$$\begin{aligned} 0 \rightarrow \text{Inn}(CoBar^\bullet(S(V))) \rightarrow \text{Der}(CoBar^\bullet(S(V))) \rightarrow \\ \rightarrow \text{Der}(CoBar^\bullet(S(V)))/\text{Inn}(CoBar^\bullet(S(V))) \rightarrow 0 \end{aligned} \quad (17)$$

As algebra,  $CoBar^\bullet(S(V))$  is free, therefore,  $\text{Inn}(CoBar^\bullet(S(V))) = CoBar^\bullet(S(V))/\mathbb{C}$  is acyclic. From the long exact sequence associated with (17) one has  $H^\bullet(\text{Der}(CoBar^\bullet(S(V)))/\text{Inn}(CoBar^\bullet(S(V)))) = H^\bullet(CoBar^\bullet(S(V)))$ , and the latter is Hochschild cohomology modulo constants of the algebra  $S(V^*)$  by the Stasheff's construction (see Section 1.1). This cohomology is equal to polyvector fields on  $V$  by the Hochschild-Kostant-Rosenberg theorem.

This example was surprising for the author, because at first look one has the following "proof" of the statement for general, not only cocomplete, coalgebra  $B$ . Consider the Hochschild complex of  $CoBar^\bullet(B)$  as a bicomplex. It has two differentials: the horizontal one is the cobar-differential, and the vertical one is the Hochschild differential. The bicomplex is placed in the I and II quarters. The spectral sequence which firstly computes the vertical (Hochschild) differential converges. Compute firstly the cohomology with respect to the Hochschild (vertical) differential. As the coalgebra  $CoBar^\bullet(B)$  is free, it has Hochschild cohomology only in degrees 0 and 1; in degree 0 it is  $\mathbb{C}$ , and in degree 1 it is  $\text{Der}(CoBar^\bullet(B))/\text{Inn}(CoBar^\bullet(B))$ . Only the terms  $E_1^{0,0}$  and  $E_1^{1,q}$ ,  $q \geq 0$  are nonzero. The spectral sequence collapses at the term  $E_2$ . This completes the "proof".

A solution to this contradiction was found in discussions with Bernhard Keller. If  $A^\bullet$  is a dg associative algebra with infinitely many nonzero grading components, there are two possible definitions of the Hochschild cohomological complex of  $A^\bullet$ . Namely, the

Hochschild complex of  $A^\bullet$  is a bicomplex, and we can take the *sum* total complex and the *product* total complex. Suppose we take the product total complex, then there is *only one* filtration of this product total complex. Namely, it is the filtration by subcomplexes  $F_k(Tot^\bullet) = \{K^{p,q}, q \geq k\}$ . This filtration satisfies the condition  $\cup_k F_k(Tot^\bullet) = Tot^\bullet$ . Another one,  $G_k(Tot^\bullet) = \{K^{p,q}, p \geq k\}$  is a filtration only on the sum total complex, that is, for the product total complex it does *not* satisfy  $\cup_k G_k(Tot^\bullet) = Tot^\bullet$ . We actually use the filtration  $G_k$  to have the (vertical) Hochschild differential in the term  $E_0$ . We see that this speculation fails, because  $G_k$  is not a filtration of the product total complex. Well, then we can take the sum total complex. But why we are guaranteed that  $Hoch^\bullet(Cobar^\bullet(S(V)))$  has 0 cohomology? We use the fact that the Hochschild cohomology of quasi-isomorphic dg algebras are the same, and  $CoBar^\bullet(S(V))$  is quasi-isomorphic to 0 because the coalgebra  $S(V)$  has a counit (see Section 1.3). But this fact is proven only for the product total complex definition of the Hochschild cohomological complex. Moreover, our discussion shows that this fact is not true for the sum total complex, and our "proof" fails either.

## 2 Applications to deformation quantization

### 2.1 A lemma

We start with the following lemma, which is a formal version of the semi-continuity of cohomology of a complex depending on a parameter, which says that in a "singular value" of the parameter the cohomology may only raise:

**Lemma.** *Let  $\mathcal{R}^\bullet$  be a  $\mathbb{Z}_{\leq 0}$ -graded complex with differential  $d_0$ , such that  $H^i(\mathcal{R}^\bullet)$  vanishes for all  $i \neq 0$ .*

- (i) *Consider  $\mathcal{R}_\hbar^\bullet = \mathcal{R}^\bullet \otimes \mathbb{C}[[\hbar]]$ . Let  $d_\hbar: \mathcal{R}_\hbar^\bullet \rightarrow \hbar \mathcal{R}_\hbar^{\bullet+1}$  be an  $\hbar$ -linear map of degree +1  $d_\hbar = d_0 + \hbar d_1 + \hbar^2 d_2 + \dots \hbar^n d_n$  (a finite sum) such that*

$$d_\hbar^2 = 0$$

*Denote by  $H_\hbar^\bullet$  the cohomology of the complex  $(\mathcal{R}_\hbar^\bullet, d_\hbar)$ . Consider the filtration  $\mathcal{R}_\hbar^\bullet \supset \hbar \mathcal{R}_\hbar^\bullet \supset \hbar^2 \mathcal{R}_\hbar^\bullet \supset \dots$ , and the induced filtration on  $H_\hbar^\bullet$ :  $F_i H_\hbar^\bullet = \text{Im}(i: H^\bullet(\hbar^i \mathcal{R}_\hbar^\bullet, d_\hbar) \rightarrow H^\bullet(\mathcal{R}_\hbar^\bullet, d_\hbar))$ . Then there are canonical isomorphisms of  $\mathbb{C}[[\hbar]]$ -modules*

$$F_i H_\hbar^0 / F_{i+1} H_\hbar^0 \xrightarrow{\sim} \hbar^i H^0(\mathcal{R}^\bullet, d_0)$$

*and  $F_i H_\hbar^k / F_{i+1} H_\hbar^k = 0$  for  $k < 0$ ,*

- (ii) *The same statement for  $\mathcal{R}_{[[\hbar]]}^\bullet = \mathcal{R}^\bullet \otimes \mathbb{C}[[\hbar]]$ , and  $d_\hbar = d_0 + \hbar d_1 + \hbar^2 d_2 + \dots$ , possibly an infinite sum,  $d_\hbar^2 = 0$ .*

*Proof.* Before a rigorous proof, let us make a remark. Consider the filtration

$$\mathcal{R}_h^\bullet \supset \hbar \mathcal{R}_h^\bullet \supset \hbar^2 \mathcal{R}_h \supset \dots$$

of the complex  $\mathcal{R}_h^\bullet$  with the differential  $d_h$ . Let us try to compute the cohomology of  $\mathcal{R}_h^\bullet$  by the spectral sequence corresponding to this filtration. The term  $E_0^{p,q} = \hbar^p \mathcal{R}_h^{p+q} / \hbar^{p+1} \mathcal{R}_h^{p+q}$ , and  $d_i, i > 0$  act by 0 on  $E_0^{p,q}$ . Therefore, the cohomology in this term is the cohomology of the differential  $d_0$  and is  $E_1^{p,-p} = \hbar^p H^0(\mathcal{R}^\bullet, d_0) / \hbar^{p+1} H^0(\mathcal{R}^\bullet, d_0)$  and  $E_1^{p,q} = 0$  for  $q \neq -p$ . Therefore, the spectral sequence collapses at the term  $E_1$  by the dimensional reasons. However, we are not guaranteed that the spectral sequence converges to the associated graded space with respect to the induced filtration on cohomology: the spectral sequence "lives" in the III-rd quarter, and it does not follow from "dimensional reasons". It is not a rigorous proof, but it somehow explains the situation.

Now we pass to a rigorous proof. We prove the both statements (i) and (ii) simultaneously.

Consider the short exact sequences of complexes  $S_k$ :

$$0 \rightarrow \hbar^{k+1} \mathcal{R}_h^\bullet \rightarrow \hbar^k \mathcal{R}_h^\bullet \rightarrow \hbar^k \mathcal{R}_h^\bullet / \hbar^{k+1} \mathcal{R}_h^\bullet \rightarrow 0 \quad (18)$$

The complex  $\hbar^k \mathcal{R}_h^\bullet / \hbar^{k+1} \mathcal{R}_h^\bullet$  has the differential  $d_0$  because all higher differentials vanish. Therefore, in the long exact sequence in cohomology corresponding to  $S_k$  we have many zero spaces, namely,  $H^\ell(\hbar^k \mathcal{R}_h^\bullet / \hbar^{k+1} \mathcal{R}_h^\bullet)$  for  $\ell \leq -1$ . Then the long exact sequence proves that the imbedding  $\hbar^{k+1} \mathcal{R}_h^\bullet \hookrightarrow \hbar^k \mathcal{R}_h^\bullet$  induces an isomorphism on  $\ell$ -th cohomology for all  $\ell \leq -1$ . Consider the end fragment of the long exact sequence:

$$\begin{aligned} \dots &\rightarrow H^1(\hbar^{k+1} \mathcal{R}_h^\bullet) \rightarrow H^1(\hbar^k \mathcal{R}_h^\bullet) \rightarrow 0 \rightarrow \\ &\rightarrow H^0(\hbar^{k+1} \mathcal{R}_h^\bullet) \rightarrow H^0(\hbar^k \mathcal{R}_h^\bullet) \rightarrow H^0(\hbar^k \mathcal{R}_h^\bullet / \hbar^{k+1} \mathcal{R}_h^\bullet) \rightarrow 0 \end{aligned} \quad (19)$$

which proves all assertions of the lemma.

Lemma is proven.  $\square$

*Remark.* Contrary with the case of the lemma, consider the case when the complex  $(\mathcal{R}^\bullet, d_0)$  is  $\mathbb{Z}_{\geq 0}$ -graded, and again only  $H^0(\mathcal{R}, d_0) \neq 0$ . Then the case (i) of lemma fails, and only (ii) is true. The proof goes as follows:

By the long exact sequence arguments one needs to prove only the surjectivity of the map  $H^0(\hbar^k \mathcal{R}_{[[\hbar]]}^\bullet) \rightarrow H^0(\hbar^k \mathcal{R}_{[[\hbar]]}^\bullet / \hbar^{k+1} \mathcal{R}_{[[\hbar]]}^\bullet)$ . The proof (which is true only over  $\mathbb{C}[[\hbar]]$ ) goes as follows:

Let  $x \in H^0(\mathcal{R}^\bullet, d_0)$  be a cycle. One needs to construct a cycle of the form  $x + \hbar x_1 + \hbar^2 x_2 + \dots$  in  $H^0(\mathcal{R}_{[[\hbar]]}^\bullet, d_h)$ . We are looking for

$$x^{(k)} = x + \hbar x_1 + \hbar^2 x_2 + \dots + \hbar^k x_k$$

such that

$$d_{\hbar}x^{(k)} = 0 \bmod \hbar^{k+1}$$

Let us note that in this situation one has  $d_0((d_{\hbar}x^{(k)})_{k+1}) = 0$ . Indeed, it follows from the fact that  $d_{\hbar}^2(x^{(k)}) = 0$ .

Now we make a step of induction. Consider  $(d_{\hbar}(x^{(k)}))_{k+1}$ . It is  $d_0$ -cycle, find  $x_{k+1} \in \mathcal{R}^0$  such that  $d_0(x_{k+1}) = (d_{\hbar}x^{(k)})_{k+1}$ . Set

$$x^{(k+1)} = x + \hbar x_1 + \dots + \hbar^{k+1} x_{k+1}$$

Then it is clear that  $d_{\hbar}(x^{(k+1)}) = 0 \bmod \hbar^{k+2}$ .

## 2.2 A proof of the classical Poincaré-Birkhoff-Witt theorem

Let  $\mathfrak{g}$  be a Lie algebra. Its universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  is defined as the quotient-algebra of the tensor algebra  $T(\mathfrak{g})$  by the two-sided ideal generated by elements  $a \otimes b - b \otimes a - [a, b]$  for any  $a, b \in \mathfrak{g}$ . The Poincaré-Birkhoff-Witt theorem says that  $\mathcal{U}(\mathfrak{g})$  is isomorphic to  $S(\mathfrak{g})$  as a  $\mathfrak{g}$ -module. We suggest here a (probably new) proof of this classical theorem, which certainly is not the simplest one, but sheds some light on the cohomological nature of the theorem.

Before starting with the proof, let us make some remark. Let us generalize the universal enveloping algebra as follows. Consider the tensor algebra  $T(x_1, \dots, x_n)$  and its quotient  $A_{c_{ij}^k}$  by the two-sided ideal generated by the relations  $x_i \otimes x_j - x_j \otimes x_i - \sum_k c_{ij}^k x_k$ ,  $1 \leq i < j \leq n$ , where  $c_{ij}^k$  are not supposed to satisfy the Jacobi identity

$$\sum_a (c_{ij}^a c_{ak}^b + c_{jk}^a c_{ai}^b + c_{ki}^a c_{aj}^b) = 0 \quad (20)$$

Then, if (20) is not satisfied, the algebra  $A_{c_{ij}^k}$  is smaller than  $S(x_1, \dots, x_n)$ , that is, the two-sided ideal, generated by the relations, is bigger than in the Lie algebra case when (20) is satisfied.

Now we pass to the proof. Let  $\mathfrak{g}$  be a Lie algebra. By the discussion in Section 1.3,  $CoBar^{\bullet}(\Lambda^+(\mathfrak{g}))$  is a free resolution of the symmetric algebra  $S(\mathfrak{g})$ . Denote the cobar-differential by  $d_0$ . Introduce in  $CoBar_{\hbar}^{\bullet} = CoBar^{\bullet}(\Lambda^+(\mathfrak{g})) \otimes \mathbb{C}[\hbar]$  a new differential  $d_0 + d_1$ , where  $d_1: CoBar_{\hbar}^{\bullet} \rightarrow \hbar \cdot CoBar_{\hbar}^{\bullet+1}$  comes from the chain differential in the Lie homology complex  $\partial: \Lambda^i(\mathfrak{g}) \rightarrow \Lambda^{i-1}(\mathfrak{g})$ . We denote

$$d_1 = \hbar \partial$$

The equation  $(d_0 + d_1)^2 = 0$  follows from the fact that the chain Lie algebra complex is a dg coalgebra, and, therefore, its cobar-complex is well-defined.

Now, by Lemma 2.1(i), the complex  $CoBar_{\hbar}^{\bullet}$  has only 0 degree cohomology, which is isomorphic to  $H^0(CoBar^{\bullet}(\Lambda^+(\mathfrak{g}))) \otimes \mathbb{C}[\hbar] = S(\mathfrak{g}) \otimes \mathbb{C}[\hbar]$  as a (filtered) vector space. On

the other hand, we can compute 0-th cohomology of  $(CoBar_{\hbar}^{\bullet}(\Lambda^+(\mathfrak{g})), d_0 + d_1)$  directly. It is the quotient of the tensor algebra  $T(\mathfrak{g}) \otimes \mathbb{C}[\hbar]$  by the two-sided ideal generated by the relations  $a \otimes b - b \otimes a - \hbar[a, b]$ ,  $a, b \in \mathfrak{g}$ .

The specialization of the last isomorphism for  $\hbar = 1$  gives the Poincaré-Birkhoff-Witt theorem.

*Remark.* It was a remark of Victor Ginzburg that for a correct "specialization at  $\hbar=1$ " we need Lemma 2.1 over polynomials, that is the case (i) of this Lemma.

### 2.3

Consider the following sequence of maps:

$$\begin{aligned} T_{poly}(V^*) &\xrightarrow{\mathcal{U}_S} \text{Hoch}(S(V)) \simeq \text{Der}(CoBar(S(V^*))) \xrightarrow{\Phi_1} \\ &\xrightarrow{\Phi_1} \text{Der}(CoBar(S^+(V^*))) / \text{Inn}(CoBar(S^+(V^*))) \end{aligned} \quad (21)$$

Here the first map is the Kontsevich formality  $L_{\infty}$  morphism for the algebra  $S(V)$ , the second isomorphism follows from the Stasheff's construction, and the third map is the map  $\Phi_1$  defined in Section 1.4.

Apply now the composition (21) to the vector space  $V[1]$  instead of  $V^*$ .

**Lemma.** *Let  $V$  be a finite-dimensional vector space. Then there is a canonical isomorphism of the graded Lie algebras  $T_{poly}(V^*) \simeq T_{poly}(V[1])$ .*

*Proof.* It is straightforward. The map maps  $k$ -polyvector field with constant coefficients on  $V^*$  to a  $k$ -linear function on  $V[1]$ , and so on.  $\square$

*Remark.* The algebras  $S(V)$  and  $\Lambda(V^*[-1])$  are Koszul dual, and they have isomorphic Hochschild comology with all structures (see [Kel]).

Denote by  $K$  the correspondence  $K: T_{poly}(V^*) \rightarrow T_{poly}(V[1])$  from Lemma. Let  $\alpha$  be a polynomial Poisson bivector on the space  $V^*$ . By the correspondence from Lemma, we get a polyvector field  $K(\alpha)$  which in general is not a bivector, but still satisfies the Maurer-Cartan equation

$$[K(\alpha), K(\alpha)] = 0 \quad (22)$$

Let us rewrite (21) for  $V[1]$ :

$$\begin{aligned} T_{poly}(V[1]) &\xrightarrow{\mathcal{U}_{\Lambda}} \text{Hoch}(\Lambda(V^*)) \simeq \text{Der}(CoBar(\Lambda(V))) \xrightarrow{\Phi_1} \\ &\xrightarrow{\Phi_1} \text{Der}(CoBar(\Lambda^-(V))) / \text{Inn}(CoBar(\Lambda^-(V))) \end{aligned} \quad (23)$$

The composition (23) maps the polyvector  $\hbar K(\alpha)$  to a derivation  $d_{\hbar}$  of degree  $+1$  in  $\text{Der}(CoBar(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]$ , which satisfies the Maurer-Cartan equation

$$(d + d_{\hbar})^2 = 0 \quad (24)$$

in  $\text{Der}/\text{Inn}$ , where  $d$  is the cobar-differential.

Actually, (24) is satisfied in  $\text{Der}(\text{CoBar}(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]$ , not only in  $\text{Der}(\text{CoBar}(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]/\text{Inn}(\text{CoBar}(\Lambda^-(V))) \otimes \mathbb{C}[[\hbar]]$ . Indeed, we suppose that  $V$  is placed in degree 0, then  $\text{CoBar}(\Lambda^-(V))$  is  $\mathbb{Z}_{\leq 0}$ -graded. Therefore, any inner derivation has degree  $\leq 0$ , while  $d_{\hbar}$  has degree  $+1$ . We have the following

## 2.4

**Lemma.** *Let  $\alpha$  be a Poisson bivector on  $V^*$ , and let  $K(\alpha)$  be the corresponding Maurer-Cartan polyvector of degree 1 in  $T_{\text{poly}}(V[1])$ . Then (23) defines an  $\hbar$ -linear derivation  $d_{\hbar}$  of  $\text{CoBar}(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]]$  of degree  $+1$  corresponding to  $\hbar K(\alpha)$ , such that*

$$(d + d_{\hbar})^2 = 0$$

where  $d$  is the cobar-differential. Moreover,  $d_{\hbar}$  obeys

$$d_{\hbar}(\xi_i \wedge \xi_j) = \hbar \text{Sym}(\alpha_{ij}) + \mathcal{O}(\hbar^2) \quad (25)$$

where  $\xi_i \wedge \xi_j \in \Lambda^2(V)$ ,  $\text{Sym}(\alpha_{ij}) \in T(V)$  is the symmetrization, and  $\{\xi_i\}$  is the basis in  $V[1]$  dual to the basis  $\{x_i\}$  in  $V^*$  in which  $\alpha = \sum_{ij} \alpha_{ij} \partial_i \wedge \partial_j$

*Proof.* We only need to prove (25), all other statements are already proven. We prove it in details in Section 2.7.  $\square$

## 2.5

Let  $A$  be an  $\hbar$ -linear associative algebra which is the quotient of the tensor algebra  $T(V) \otimes \mathbb{C}[[\hbar]]$  of a vector space  $V$  by the two-sided ideal generated by relations

$$x \otimes y - y \otimes x = R(x, y)$$

for any  $x, y \in V$ , where  $R(x, y) \in \hbar T(V) \otimes \mathbb{C}[[\hbar]]$ . Consider the following filtration:

$$A \supset \hbar A \supset \hbar^2 A \supset \hbar^3 A \supset \dots \quad (26)$$

This is clearly an algebra filtration:  $(\hbar^k A) \cdot (\hbar^\ell A) \subset \hbar^{k+\ell} A$ . Consider the associated graded algebra  $\text{gr}A$ . We say that the algebra  $A$  is a Poincaré-Birkhoff-Witt (PBW) algebra if  $\text{gr}A \simeq S(V) \otimes \mathbb{C}[[\hbar]]$  as a graded  $\mathbb{C}[[\hbar]]$ -linear algebra.

In general,  $\text{gr}A$  is less than  $S(V) \otimes \mathbb{C}[[\hbar]]$ , it is a quotient of  $S(V) \otimes \mathbb{C}[[\hbar]]$ . One can say that the PBW property is equivalent to the property that the quotient-algebra has "the maximal possible size".

## 2.6

Let  $\alpha$  be a polynomial Poisson bivector in  $V^*$ . In Sections 2.3 and 2.4 we constructed an  $\hbar$ -linear derivation  $d_\hbar$  on  $CoBar(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]]$  such that  $(d + d_\hbar)^2 = 0$  where  $d$  is the cobar-differential. By Section 1.3, the cobar-complex  $CoBar(\Lambda^-(V))$  is a free resolution of the algebra  $S(V)$ , in particular, the cohomology of  $d$  does not vanish only in degree 0 where it is equal to  $S(V)$ . We are in the situation of Lemma 2.1. In particular, the dg algebra  $(CoBar(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]], d + d_\hbar)$  has only non-vanishing cohomology in degree 0, and this 0-degree cohomology is an algebra, which is a PBW algebra by Lemma 2.1.

**Theorem.** *The construction above constructs from a Poisson polynomial bivector  $\alpha$  on  $V^*$  an algebra  $A_\alpha$  with generators  $x_1, \dots, x_n$  and relations  $[x_i, x_j] = d_\hbar(\xi_i \wedge \xi_j)$ . This algebra is a PBW algebra.  $\square$*

**Conjecture.** *The algebra  $A_\alpha$  is isomorphic to the Kontsevich star-algebra on  $S(V) \otimes \mathbb{C}[[\hbar]]$  constructed from the Poisson bivector  $\alpha$ . (We suppose that in the formality morphisms  $\mathcal{U}_\Lambda: T_{poly}(V[1]) \rightarrow \text{Hoch}(\Lambda(V^*))$  in (23), and  $\mathcal{U}_S: T_{poly}(V^*) \rightarrow \text{Hoch}(S(V))$  which is used in the construction of the star-product, one uses the same propagator in the definition of the Kontsevich integrals, see [K97]). The isomorphism is given by the symmetrization map.*

## 2.7 An explicit formula

One can write down explicitly the relations in the algebra  $A_\alpha$ , in the terms of the Kontsevich integrals [K97]. For this we need to find explicit formula for the  $\hbar$ -linear derivation  $d_\hbar$  in  $CoBar(\Lambda^-(V)) \otimes \mathbb{C}[[\hbar]]$ . Here we suppose some familiarity with [K97].

First of all, recall how the Kontsevich deformation quantization formula is written. Let  $\alpha$  be a Poisson structure on  $V^*$ . Then the formula is

$$f \star g = f \cdot g + \sum_{k \geq 1} \hbar^k \left( \sum_{m \geq 1} \frac{1}{m!} \sum_{\Gamma \in G_{2,m}^2} W_\Gamma U_\Gamma(\alpha, \dots, \alpha) \right) \quad (27)$$

Here  $\Gamma$  is an admissible graph with two vertices on the "real line" and  $m$  vertices in the upper half-plane, and with 2 outgoing edges at each vertex in the upper half-plane, that is,  $\Gamma \in G_{2,m}^2$ , in particular, it is an oriented graph with  $2m$  edges;  $W_\Gamma$  is the Kontsevich integral of the graph  $\Gamma$ . *Let us note that the all graphs involved in (27) may have arbitrary many incoming edges at each vertex at the upper half-plane, and exactly two outgoing edges.*

Now let  $\alpha = \sum_{ij} \alpha_{ij} \partial_i \wedge \partial_j$ , where  $\alpha_{ij} = \sum_I c_{ij}^I x_{i_1} \dots x_{i_k}$  ( $I$  is a multi-index).

Then the "Koszul dual" polyvector  $K(\alpha)$  is a polyvector field with quadratic coefficients:

$$K(\alpha) = \sum_{i,j,I} c_{ij}^I (\xi_i \xi_j) \cdot \partial_{\xi_{i_1}} \wedge \dots \wedge \partial_{\xi_{i_k}} \quad (28)$$

It has total degree 1 and satisfies the Maurer-Cartan equation.

Firstly we write the formula for the image of  $K(\hbar\alpha)$  by the Kontsevich formality, that is, denote by

$$\mathcal{U}(K(\alpha)) = \hbar U_1(K(\alpha)) + \hbar^2 \frac{1}{2} U_2(K(\alpha), K(\alpha)) + \dots + \hbar^k \frac{1}{k!} U_k(K(\alpha), \dots, K(\alpha)) + \dots \quad (29)$$

We can write down explicitly this formula in graphs. *Let us note the the graphs involving in (29) may have arbitrary many outgoing edges in the vertices at the upper half-plane, but exactly two incoming edges, because all components of  $K(\alpha)$  are quadratic polyvector fiels*, and by a simple dimension computation. That is, in a sense the graphs in (29) are "dual" to the graphs in (27).

Let us note also that the right-hand side of (29) is a polydifferential operator in  $\text{Hoch}(\Lambda(V^*))$  of non-homogeneous Hochschild degree, but of the total (Hochschild degree and  $\Lambda$ -degree)  $+1$ .

Now we should apply to  $\mathcal{U}(K)$  our map  $\Phi_1$  to get a derivation of the cobar-complex  $\text{CoBar}(\Lambda^-(V))$ . After this, we get the final answer for  $d_{\hbar}$ .

Let us compute its component in the first power of  $\hbar$ . It is just the Hochschild-Kostant-Rosenberg map of  $U_1(K(\alpha))$  which is the symmetrization map  $\text{Sym}: S(V) \rightarrow T(V)$  in this case.

Let us note that in the case of a quadratic Poisson structure algebra the relations  $R_{ij} \in (V \otimes V) \otimes \mathbb{C}[[\hbar]] \subset T(V) \otimes \mathbb{C}[[\hbar]]$  are quadratic.

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